

MODULE THEORY OVER LEAVITT PATH ALGEBRAS AND K -THEORY

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ABSTRACT. Let k be a field and let E be a finite quiver. We study the structure of the finitely presented modules of finite length over the Leavitt path algebra $L_k(E)$ and show its close relationship with the finite-dimensional representations of the inverse quiver \overline{E} of E , as well as with the class of finitely generated $P_k(E)$ -modules M such that $\mathrm{Tor}_q^{P_k(E)}(k^{|E^0|}, M) = 0$ for all q , where $P_k(E)$ is the usual path algebra of E . By using these results we compute the higher K -theory of the von Neumann regular algebra $Q_k(E) = L_k(E)\Sigma^{-1}$, where Σ is the set of all square matrices over $P_k(E)$ which are sent to invertible matrices by the augmentation map $\epsilon: P_k(E) \rightarrow k^{|E^0|}$.

1. INTRODUCTION

For a field k and an integer $n \geq 2$, the Leavitt algebra $L(1, n)$ of type $(1, n)$ is the algebra with generators x_i, y_j , $1 \leq i, j \leq n$ with defining relations given by

$$(x_1, \dots, x_n)(y_1, \dots, y_n)^t = 1, \quad (y_1, \dots, y_n)^t(x_1, \dots, x_n) = I_n,$$

where I_n is the $n \times n$ identity matrix. These algebras, first studied by Leavitt in [20] and [21], provide universal examples of algebras without the invariant basis number property: observe that right multiplication by the row (x_1, \dots, x_n) gives an isomorphism from the free left $L(1, n)$ -module of rank one onto the free left $L(1, n)$ -module of rank n . They are algebraic analogues of the *Cuntz algebras* \mathcal{O}_n , introduced independently by Cuntz in [14]. The first author analyzed in [3] the structure of the finitely presented modules over $L(1, n)$ in connection with the structure of certain classes of finitely presented modules over the free algebras $k\langle x_1, \dots, x_n \rangle$ and $k\langle y_1, \dots, y_n \rangle$. Both free algebras embed in $L(1, n)$, and the abelian category \mathcal{S} of finitely presented left $L(1, n)$ -modules of finite length is equivalent to a quotient category of the abelian category of finite-dimensional $k\langle y_1, \dots, y_n \rangle$ -modules by a certain Serre subcategory, see [3, Theorem 5.1]. Let Σ be the class of all the square matrices over $k\langle x_1, \dots, x_n \rangle$ that are sent to an invertible matrix by the augmentation map. Then \mathcal{S} is identified with the category of finitely presented

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Σ -torsion modules in [3, Theorem 6.2], and this is used to give a formula for $K_1(Q_n)$, where $Q_n = L(1, n)\Sigma^{-1}$ is the universal localization of $L(1, n)$ with respect to Σ , which was shown in [8] to be a simple von Neumann regular ring.

The main purpose of this paper is to generalize these results to the much wider context of path algebras. Our main guiding principle in tackling this problem is the idea that free algebras are prototypical examples of path algebras, and many results on free algebras should admit suitable generalizations to this setting. For each finite (or even row-finite) quiver E , there is a *Leavitt path algebra* $L_k(E)$, described below, which plays a similar role with respect to the usual path algebra $P_k(E)$ as $L(1, n)$ does with respect to the free algebra $k\langle x_1, \dots, x_n \rangle$. (Recall that $k\langle x_1, \dots, x_n \rangle$ is the path algebra of the quiver with one vertex and n arrows.) The Leavitt path algebras $L_k(E)$ were first introduced in [1] and [9], and have been intensively studied by various authors since then. The *regular algebra of E* , denoted by $Q_k(E)$, was constructed in [4], and is the natural generalization of the algebra Q_n described above; see below for the definition. It follows from [4, Theorem 4.2] that $K_0(Q_k(E)) \cong K_0(L_k(E))$ for every finite quiver E . We will compute here (Theorem 7.5) all the higher K -theory groups of $Q_k(E)$ in terms of the K -theory groups of $L_k(E)$, recently computed in [6], and the K -theory of a certain abelian category $\mathcal{B}la(P(E))$ of objects of finite length. This is new even for the regular algebra Q_n of the classical Leavitt algebra $L(1, n)$, since only K_1 was considered in [3].

Unless otherwise is stated all modules are left modules. In the following, k will denote a field and $E = (E^0, E^1, r, s)$ a finite quiver (oriented graph) with $E^0 = \{1, \dots, d\}$. Here $s(e)$ is the *source vertex* of the arrow e , and $r(e)$ is the *range vertex* of e . A *path* in E is either an ordered sequence of arrows $\alpha = e_1 \cdots e_n$ with $r(e_t) = s(e_{t+1})$ for $1 \leq t < n$, or a path of length 0 corresponding to a vertex $i \in E^0$, which will be denoted by p_i . The paths p_i are called *trivial paths*, and we have $r(p_i) = s(p_i) = i$. A non-trivial path $\alpha = e_1 \cdots e_n$ has length n and we define $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. We will denote the length of a path α by $|\alpha|$, the set of all paths of length n by E^n (for $n > 1$), and the set of all paths by E^* .

Let us recall the construction of the Leavitt path algebra $L(E) = L_k(E)$ and of the regular algebra $Q(E) = Q_k(E)$ of a quiver E . These algebras fit into the following all-important commutative diagram of injective algebra morphisms:

$$(1.1) \quad \begin{array}{ccccccc} k^d & \longrightarrow & P(E) & \xrightarrow{\iota_\Sigma} & P_{\text{rat}}(E) & \longrightarrow & P((E)) \\ \downarrow & & \downarrow \iota_{\Sigma_1} & & \downarrow \iota_{\Sigma_1} & & \downarrow \iota_{\Sigma_1} \\ P(\overline{E}) & \xrightarrow{\iota_{\Sigma_2}} & L(E) & \xrightarrow{\iota_\Sigma} & Q(E) & \longrightarrow & U(E) \end{array}$$

Here $P(E)$ is the path k -algebra of E , \overline{E} denotes the inverse quiver of E , that is, the quiver obtained by changing the orientation of all the arrows in E , $P((E))$ is the algebra of formal power series on E , and $P_{\text{rat}}(E)$ is the algebra of rational series, which is by

definition the division closure of $P(E)$ in $P((E))$ (which agrees with the rational closure, see [4, Observation 1.18]). The maps ι_Σ and ι_{Σ_i} indicate universal localizations with respect to the sets Σ and Σ_i respectively. Here Σ is the set of all square matrices over $P(E)$ that are sent to invertible matrices by the augmentation map $\epsilon: P(E) \rightarrow k^{|E^0|}$, which coincides with the set of square matrices over $P(E)$ which are invertible over $P((E))$ ([4, Observation 1.19]). By [4, Theorem 1.20], the algebra $P_{\text{rat}}(E)$ coincides with the universal localization $P(E)\Sigma^{-1}$. The set $\Sigma_1 = \{\mu_i \mid i \in E^0, s^{-1}(i) \neq \emptyset\}$ is the set of morphisms between finitely generated projective left $P(E)$ -modules defined by

$$\begin{aligned} \mu_i: P(E)p_i &\longrightarrow \bigoplus_{j=1}^{n_i} P(E)p_{r(e_j^i)} \\ r &\longmapsto (re_1^i, \dots, re_{n_i}^i) \end{aligned}$$

for any $i \in E^0$ such that $s^{-1}(i) \neq \emptyset$, where $s^{-1}(i) = \{e_1^i, \dots, e_{n_i}^i\}$. By a slight abuse of notation, we use also μ_i to denote the corresponding maps between finitely generated projective left $P_{\text{rat}}(E)$ -modules and $P((E))$ -modules respectively. The set $\Sigma_2 = \{\nu_i \mid i \in E^0, s^{-1}(i) \neq \emptyset\}$ is the set of morphisms between finitely generated projective left $P(\overline{E})$ -modules defined by

$$\begin{aligned} \nu_i: \bigoplus_{j=1}^{n_i} P(\overline{E})p_{r(e_j^i)} &\longrightarrow P(\overline{E})p_i \\ (r_1, \dots, r_{n_i}) &\longmapsto \sum_{j=1}^{n_i} r_j \bar{e}_j^i. \end{aligned}$$

for each $i \in E^0$ such that $s^{-1}(i) \neq \emptyset$.

The following relations hold in $Q(E)$:

- (V) $p_v p_{v'} = \delta_{v,v'} p_v$ for all $v, v' \in E^0$.
- (E1) $p_{s(e)} e = e p_{r(e)} = e$ for all $e \in E^1$.
- (E2) $p_{r(e)} \bar{e} = \bar{e} p_{s(e)} = \bar{e}$ for all $e \in E^1$.
- (CK1) $\bar{e} e' = \delta_{e,e'} p_{r(e)}$ for all $e, e' \in E^1$.
- (CK2) $p_v = \sum_{\{e \in E^1 \mid s(e)=v\}} e \bar{e}$ for every $v \in E^0$ that emits edges.

The Leavitt path algebra $L(E) = P(E)\Sigma_1^{-1} = P(\overline{E})\Sigma_2^{-1}$ is the algebra generated by $\{p_v \mid v \in E^0\} \cup \{e, \bar{e} \mid e \in E^1\}$ subject to the relations (V)–(CK2) above; see for instance [1] and [9]. Relations (CK1) and (CK2) are called the *Cuntz-Krieger relations*, see [15]. By [4, Theorem 4.2], the algebra $Q(E)$ is a von Neumann regular hereditary ring and $Q(E) = P(E)(\Sigma \cup \Sigma_1)^{-1}$.

A *sink* in E is a vertex $i \in E^0$ such that $s^{-1}(i) = \emptyset$, that is, i does not emit any arrow. The set of sinks of E will be denoted by $\text{Sink}(E)$. With this terminology we can summarize the results on the K -theory of the Leavitt algebra $L_k(E)$, obtained in [6], as follows. Consider the adjacency matrix $A_E = (a_{ij}) \in \mathbb{Z}^{(E^0 \times E^0)}$, $a_{ij} = \#\{\text{arrows from } i \text{ to } j\}$. Write N_E and 1 for the matrices in $\mathbb{Z}^{(E^0 \times E^0 \setminus \text{Sink}(E))}$ which result from A_E^t

and from the identity matrix after removing the columns corresponding to sinks. Then there is a long exact sequence ($n \in \mathbb{Z}$)

$$\dots \rightarrow K_n(k)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1-N_E} K_n(k)^{(E_0)} \longrightarrow K_n(L_k(E)) \longrightarrow K_{n-1}(k)^{(E_0 \setminus \text{Sink}(E))}.$$

In particular

$$K_0(L_k(E)) \cong \text{coker}(1 - N_E: \mathbb{Z}^{(E_0 \setminus \text{Sink}(E))} \longrightarrow \mathbb{Z}^{(E_0)}),$$

and

$$K_1(L_k(E)) \cong \text{coker}(1 - N_E: (k^\times)^{(E_0 \setminus \text{Sink}(E))} \longrightarrow (k^\times)^{(E_0)})$$

$$\bigoplus \ker(1 - N_E: \mathbb{Z}^{(E_0 \setminus \text{Sink}(E))} \longrightarrow \mathbb{Z}^{(E_0)}).$$

In Theorem 7.5, we show that, for $i \geq 1$,

$$K_i(Q(E)) \cong K_i(L(E)) \bigoplus \text{Bla}_{i-1}(P(E)),$$

where $\text{Bla}_*(P(E))$ is the K -theory of the abelian category $\mathcal{Bla}(P(E))$ consisting of finitely generated $P(E)$ -modules M such that $\text{Tor}_q^{P_k(E)}(k^{|E^0|}, M) = 0$ for all q . This category is shown in Proposition 7.2 to be exactly the category of finitely presented $L(E)$ -modules of finite length without nonzero projective submodules. Observe that, by the “Devissage” Theorem ([27, 5.3.24]) and the results in the present paper, the groups $\text{Bla}_i(P(E))$ are the direct sum of the K_i groups of the endomorphism rings $\text{End}_{P(\overline{E})}(M)^{\text{op}}$, where M ranges over all the finite-dimensional non-projective simple $P(\overline{E})$ -modules which are not isomorphic to one of the simple modules $\text{coker}(\nu_j)$ for $\nu_j \in \Sigma_2$.

The rest of the paper is organized as follows. As a preparation for our main results, we develop in Sections 2 and 3 some results about the structure of finitely presented modules over a path algebra. This is done by extending to this context some of the tools developed by Cohn to study firs. In particular we show in Theorem 3.14 that every finitely related $P(E)$ -module L has a projective submodule Q such that L/Q is finite-dimensional over k , generalizing a result of Lewin [22] for the free algebra. Section 4 establishes the important fact that $L(E)$ is flat as a *right* $P(\overline{E})$ -module, which will be often used afterwards. We start our study of the module theory over Leavitt path algebras in Section 5, obtaining in Proposition 5.9 a description of the finitely presented $L(E)$ -modules of finite length as induced modules from finite-dimensional $P(\overline{E})$ -modules. In Section 6, the abelian categories $\mathbf{fp}(L(E))$ and $\mathbf{fp}(L(E))_{\text{fl}}$ of finitely presented, and finitely presented $L(E)$ -modules of finite length, respectively, are shown to be equivalent to the quotient categories of the corresponding categories of $P(\overline{E})$ -modules modulo the Serre subcategory generated by the simple finite-dimensional $P(\overline{E})$ -modules $\text{coker}(\nu_j)$, for $\nu_j \in \Sigma_2$. Finally we discuss the notion of Blanchfield modules in Section 7, which we have adapted from [26], and we show that the category of finitely generated Blanchfield $P(E)$ -modules agrees with various relevant categories. In particular it is the category of torsion modules for both universal localizations $P(E) \rightarrow P(E)\Sigma^{-1}$ and $L(E) \rightarrow L(E)\Sigma^{-1}$ (Proposition 7.3), and coincides with the category of finitely presented $L(E)$ -modules of finite length without nonzero projective submodules

(Proposition 7.2). The K -theory results described above are deduced then from the long exact sequence of Neeman and Ranicki for stably flat universal localizations [24],[25],[23].

2. FINITELY PRESENTED MODULES OVER PATH ALGEBRAS

Let k be a field and let $R = k\langle X \rangle$ be the free algebra in n variables. Recall that given an R -module M of finite k -dimension we have the Lewin-Schreier formula relating $\chi_R(M)$, the Euler characteristic, with the k -dimension of M :

$$\chi_R(M) = (1 - n) \dim_k(M)$$

(see [22, Theorem 4] or [13, Theorem 2.5.3]). Using a general result due to Bergman and Dicks [11] we will see that a similar formula holds for the path algebra.

To state the formula in our situation we will need a more general context. Let R be any ring. If an R -module M has a finite resolution by finitely generated projective modules,

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

it is known that the element $\chi_R(M) := \sum (-1)^i [P_i] \in K_0(R)$ is an invariant of M called its *Euler characteristic*.

Let A be any ring. If R is an A -ring, then it makes sense to compare $\chi_A(M) \in K_0(A)$ and $\chi_R(M) \in K_0(R)$ when both are defined. We have the following definition due to Bergman and Dicks:

Definition 2.1 ([11, (64)]). An A -ring R will be called a *left Lewin-Schreier A -ring* if

- (1) every left R -module M which has a finite resolution by finitely generated projectives over A also has such a resolution over R , and
- (2) there exists a homomorphism $\lambda_R^A: K_0(A) \rightarrow K_0(R)$ such that, for such an M , $\chi_R(M) = \lambda_R^A \chi_A(M)$.

Let R be an A -ring. We will denote by $\tau_R^A: K_0(A) \rightarrow K_0(R)$ the homomorphism induced by the functor $R \otimes_A -$.

Proposition 2.2. *Let E be a finite quiver with $E^0 = \{1, \dots, d\}$. Then $P(E)$ is a left Lewin-Schreier k^d -ring with $\lambda_{P(E)}^{k^d} = (\mathbf{1} - A_E^t) \tau_{P(E)}^{k^d}$.*

Proof. We write $R = P(E)$ and $A = k^d$. Let $N \subseteq R$ be the A -bimodule generated by the edges. It is easy to check that the path algebra of a quiver is isomorphic to the tensor A -ring associated to the bimodule generated by the edges (see [10, Proposition III.1.3]). Therefore, by [11, (63)] we get the following exact sequence:

$$(2.1) \quad 0 \longrightarrow R \otimes_A N \otimes_A R \longrightarrow R \otimes_A R \longrightarrow R \longrightarrow 0.$$

Let M be a left R -module finitely generated as A -module. Applying the functor $- \otimes_R M$ to the exact sequence (2.1) we get a resolution of M by finitely generated projective left R -modules

$$0 \longrightarrow R \otimes_A N \otimes_A M \longrightarrow R \otimes_A M \longrightarrow M \longrightarrow 0,$$

and so, R satisfies the first condition in the definition.

As an A -module, M is isomorphic to $(Ap_1)^{\alpha_1} \oplus \cdots \oplus (Ap_d)^{\alpha_d}$ for some $\alpha_1, \dots, \alpha_d \in \mathbb{N}$. We put $A_E = (a_{ij})$. We have the following isomorphisms of left R -modules

$$R \otimes_A Ap_i \cong Rp_i, \quad R \otimes_A N \otimes_A Ap_i \cong \bigoplus_{j=1}^d (Rp_j)^{a_{ji}}.$$

So, we get

$$\chi_R(M) = [R \otimes_A M] - [R \otimes_A N \otimes_A M] = \sum_{i=1}^d \alpha_i [Rp_i] - \sum_{i=1}^d \sum_{j=1}^d a_{ji} \alpha_j [Rp_i] = (\mathbf{1} - A_E^t) \tau_R^A \chi_A(M)$$

as wanted. \square

3. THE WEAK ALGORITHM FOR PATH ALGEBRAS

The path algebra can be profitably thought of as a generalization of the free algebra and, quite often, properties of the latter admit a generalization to the former. In this section we generalize Cohn's weak algorithm (see [13, Chapter 2]) to the context of path algebras and prove several of its basic properties. The main result in this section is Theorem 3.14 which is a version of Lewin's Theorem (see [22, Theorem 2]) for path algebras.

Let R be a non-zero ring. Recall that a *filtration* on R is given by a map $\nu: R \rightarrow \mathbb{N} \cup \{-\infty\}$ with the following properties:

- (1) $\nu(r) \geq 0$ for all $r \neq 0$, $\nu(0) = -\infty$,
- (2) $\nu(r - s) \leq \max\{\nu(r), \nu(s)\}$,
- (3) $\nu(rs) \leq \nu(r) + \nu(s)$,
- (4) $\nu(1) = 0$.

If equality holds in (3), we have a degree function. Even in the general case we shall call $\nu(r)$ the *degree* of r . It is easy to see that the path algebra $P(E)$ is a filtered ring with respect to the degree. A filtration is also determined by the additive subgroups R_h given by the elements of degree at most h .

Let R be a ring with a filtration ν . Given an R -module M a *filtration* on M is given by a map $\mu: M \rightarrow \mathbb{N} \cup \{-\infty\}$ such that

- (1) $\mu(m) \geq 0$ for all $m \neq 0$, $\mu(0) = -\infty$,
- (2) $\mu(m - n) \leq \max\{\mu(m), \mu(n)\}$,
- (3) $\mu(mr) \leq \mu(m) + \nu(r)$.

Like in the ring case, a filtration on M is also determined by the additive subgroups M_h given by the elements of degree at most h .

The following definition is useful to generalize Cohn's concept of μ -independence to the context of path algebras.

Definition 3.1. Let (R, ν) be a filtered ring. A *set of vertices in R* is a finite set P of zero-degree, pairwise orthogonal idempotents in R such that $1 = \sum_{p \in P} p$. We also say that R has a *vertex-type decomposition given by P* .

- Examples 3.2.** 1. Any filtered ring has a trivial vertex-type decomposition given by $P = \{1\}$.
 2. The path algebra of a finite quiver E has a vertex-type decomposition given by the vertices $P = \{p_i \mid i \in E^0\}$. This is the example to bear in mind.
 3. Mixed path algebras as defined in [5] have also a vertex-type decomposition given by the vertices.

In the following definitions and results R will denote a ring with a filtration ν , $P = \{p_1, \dots, p_d\}$ will be a set of vertices in R and M will be an R -module with a filtration μ .

Definition 3.3. We say that the family $(m_i)_{i \in I} \in \prod_{i \in I} p_{n_i} M$ is *left P - μ -dependent* provided that exists a family $(r_i)_{i \in I} \in \bigoplus_{i \in I} R p_{n_i}$ such that

$$\mu \left(\sum_{i \in I} r_i m_i \right) < \max_{i \in I} \{ \nu(r_i) + \mu(m_i) \}$$

or if some $m_i = 0$. Otherwise the family $(m_i)_{i \in I}$ is said to be *left P - μ -independent*.

When $P = \{1\}$ (and $M = R$) we recover Cohn's definitions of left μ -dependent and left μ -independent family (see [13, Pag. 95]). Recall that in Cohn's setting a left μ -independent family generates a free module (because it is also a left linearly independent family). In the general case, the point is the fact that a left P - μ -independent family generates a projective module:

Proposition 3.4. *In the above situation, let $(m_i)_{i \in I} \in \prod_{i \in I} p_{n_i} M$ be a P - μ -independent family. Then the submodule $\sum_{i \in I} R m_i$ is projective.*

Proof. Indeed, by the P - μ -independence of the family, the epimorphism

$$\begin{aligned} \bigoplus_{i \in I} R p_{n_i} &\longrightarrow \sum_{i \in I} R m_i \subseteq M \\ (r_i)_{i \in I} &\longmapsto \sum_{i \in I} r_i m_i \end{aligned}$$

is an isomorphism. □

Definition 3.5. An element $m \in M$ is said to be *left P - μ -dependent* on a family $(m_i)_{i \in I} \in \prod_{i \in I} p_{n_i} M$ if either $m = 0$ or there exists a family $(r_i)_{i \in I} \in \bigoplus_{i \in I} R p_{n_i}$ such that

$$\mu \left(m - \sum_{i \in I} r_i m_i \right) < \mu(m) \quad \text{and} \quad \forall i \in I, \nu(r_i) + \mu(m_i) \leq \mu(m).$$

In the contrary case m is said to be *left P - μ -independent* of $(m_i)_{i \in I}$.

We will also need the definition of left P - μ -dependence of an element on a general set:

Definition 3.6. An element $m \in M$ is said to be *left P - μ -dependent* on a set $S \subseteq M$ provided that there exists a family $(m_i)_{i \in I} \in \prod_{i \in I} p_{n_i} S$ such that m is left P - μ -dependent on it. Otherwise m is said to be *left P - μ -independent* of S .

Now, we can generalize the weak algorithm to our framework:

Definition 3.7. We say that M satisfies the *weak algorithm* relative to μ and P if in every finite left P - μ -dependent family $(m_i)_{i=1, \dots, \ell} \in \prod_{i=1}^{\ell} p_{n_i} M$ where

$$\mu(m_1) \leq \dots \leq \mu(m_\ell),$$

some m_i is left P - μ -dependent on m_1, \dots, m_{i-1} .

Applying these definitions to the regular module $M = {}_R R$ with the filtration $\mu = \nu$ we also have these concepts defined for the filtered ring (R, ν) .

Given an expression $\sum_{i \in I} r_i m_i \in M$ with $m_i \in M$ and $r_i \in R$ we will refer to $\max_i \{\nu(r_i) + \mu(m_i)\}$ as its *formal degree*. We remark that the definition of P - μ -independence of a family states that the degree of elements represented by certain expressions should equal the formal degree of these expressions.

The previous definitions are motivated by the fact that any free module over the path algebra satisfies the weak algorithm relative to a suitable degree, as we show in our next result. This will be improved in Theorem 3.13, where it is shown that the $P(E)$ -modules satisfying the weak algorithm relative to some filtration are precisely the projective $P(E)$ -modules.

Proposition 3.8. *Let E be a finite quiver with $E^0 = \{1, \dots, d\}$. Let M be a free $P(E)$ -module freely generated by \mathcal{B} and consider a map $\mu: \mathcal{B} \rightarrow \mathbb{N}$. If we extend μ to M as the formal degree, then (M, μ) is a filtered module and satisfies the weak algorithm relative to μ and $P = \{p_1, \dots, p_d\}$, the set of vertices given by the vertices of E .*

Proof. First of all, since elements in M have a unique expression as $P(E)$ -linear combination of elements in \mathcal{B} , the formal degree gives a well defined filtration on M . Now we will prove that M satisfies the weak algorithm relative to μ and P . Let $(m_i)_{i=1, \dots, \ell} \in \prod_{i=1}^{\ell} (p_{n_i} M \setminus \{0\})$ be a left P - μ -dependent family such that $\mu(m_1) \leq \dots \leq \mu(m_\ell)$. There exists an element $(r_i)_{i=1, \dots, \ell} \in \bigoplus_{i=1}^{\ell} P(E) p_{n_i}$ such that

$$(3.1) \quad \mu \left(\sum_{i=1}^{\ell} r_i m_i \right) < t = \max_i \{\nu(r_i) + \mu(m_i)\}.$$

By omitting some terms if necessary we may assume that, for all i , $\nu(r_i) + \mu(m_i) = t$ and hence $\nu(r_\ell) \leq \dots \leq \nu(r_1)$.

Since \mathcal{B} is a basis for M , every m_i has a unique expression $m_i = \sum_{b \in \mathcal{B}} r_b^i b$. Moreover, from $p_{n_i} m_i = m_i$ we get that $p_{n_i} r_b^i = r_b^i$. Therefore,

$$\mu \left(\sum_{i=1}^{\ell} r_i m_i \right) = \mu \left(\sum_{i=1}^{\ell} r_i \left(\sum_{b \in \mathcal{B}} r_b^i b \right) \right) = \mu \left(\sum_{b \in \mathcal{B}} \left(\sum_{i=1}^{\ell} r_i r_b^i \right) b \right).$$

Let $\gamma \in \text{supp}(r_\ell)$ (the support of r_ℓ) be a path of maximal length, say t_0 . Now, given $r, s \in P(E)$, we have that

$$(3.2) \quad \delta_\gamma(sr) \equiv \delta_\gamma(s)r \pmod{P(E)_{\nu(r)-1}},$$

where δ_γ is the right transduction corresponding to γ , that is, $\delta_\gamma(\gamma\tau') = \tau'$ and $\delta_\gamma(\tau) = 0$ if τ does not start with γ ; see [4, Section 1]. This is clear if s is a monomial of length at least t_0 ; in fact we then have equality. If s is a monomial of length less than t_0 , the right-hand side of (3.2) is zero, and so it holds as a congruence. The general case follows by linearity.

Now, for all i and all b , the element $\delta_\gamma(r_i)r_b^i$ differs from $\delta_\gamma(r_i r_b^i)$ by a term of degree less than $\nu(r_b^i)$. Therefore, we have

$$\nu \left(\sum_{i=1}^{\ell} (\delta_\gamma(r_i)r_b^i - \delta_\gamma(r_i r_b^i)) \right) \leq \max_i \{ \nu(\delta_\gamma(r_i)r_b^i - \delta_\gamma(r_i r_b^i)) \} < \max_i \{ \nu(r_b^i) \}.$$

From this inequality, we get

$$\begin{aligned} (3.3) \quad & \mu \left(\sum_{b \in \mathcal{B}} \left(\sum_{i=1}^{\ell} \delta_\gamma(r_i)r_b^i \right) b - \sum_{b \in \mathcal{B}} \delta_\gamma \left(\sum_{i=1}^{\ell} r_i r_b^i \right) b \right) = \\ & = \mu \left(\sum_{b \in \mathcal{B}} \left(\sum_{i=1}^{\ell} (\delta_\gamma(r_i)r_b^i - \delta_\gamma(r_i r_b^i)) \right) b \right) \\ & = \max_{b \in \mathcal{B}} \left\{ \mu(b) + \nu \left(\sum_{i=1}^{\ell} (\delta_\gamma(r_i)r_b^i - \delta_\gamma(r_i r_b^i)) \right) \right\} \\ & < \max_{b \in \mathcal{B}} \left\{ \mu(b) + \max_i \{ \nu(r_b^i) \} \right\} \\ & = \max_{b \in \mathcal{B}} \left\{ \max_i \{ \mu(r_b^i b) \} \right\} = \max_i \left\{ \max_{b \in \mathcal{B}} \{ \mu(r_b^i b) \} \right\} \\ & = \max_i \left\{ \mu \left(\sum_{b \in \mathcal{B}} r_b^i b \right) \right\} = \max_i \{ \mu(m_i) \} = \mu(m_\ell). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 (3.4) \quad \mu \left(\sum_{b \in \mathcal{B}} \delta_\gamma \left(\sum_{i=1}^{\ell} r_i r_b^i \right) b \right) &= \max_{b \in \mathcal{B}} \left\{ \mu(b) + \nu \left(\delta_\gamma \left(\sum_{i=1}^{\ell} r_i r_b^i \right) \right) \right\} \\
 &\leq \max_{b \in \mathcal{B}} \left\{ \mu(b) + \nu \left(\sum_{i=1}^{\ell} r_i r_b^i \right) \right\} - t_0 \\
 &= \mu \left(\sum_{i=1}^{\ell} r_i m_i \right) - t_0 \\
 &< t - t_0 = \mu(m_\ell).
 \end{aligned}$$

Hence, by (3.3) and (3.4) we get that

$$\mu \left(\sum_{i=1}^{\ell} \delta_\gamma(r_i) m_i \right) = \mu \left(\sum_{b \in \mathcal{B}} \left(\sum_{i=1}^{\ell} \delta_\gamma(r_i) r_b^i \right) b \right) < \mu(m_\ell)$$

and, since $\delta_\gamma(r_\ell) \in k^\times p_{n_\ell}$ we deduce that m_ℓ is left P - μ -dependent on $m_1, \dots, m_{\ell-1}$ as wanted. \square

In particular, the path algebra $P(E)$ satisfies the weak algorithm relative to the degree and the obvious set of vertices. It is straightforward to see that the weak algorithm is inherited by submodules:

Lemma 3.9. *Let (R, ν) be a filtered ring with a set of vertices P and let (M, μ) be a filtered right R -module satisfying the weak algorithm relative to μ and P . Then every submodule $N \subseteq M$ satisfies the weak algorithm relative to $\mu|_N$ and P .*

We have the following restriction for rings with weak algorithm:

Proposition 3.10. *Let (R, ν) be a filtered ring with a set of vertices P . If R satisfies the weak algorithm relative to ν and P then R_0 is a semisimple ring.*

Proof. The set $R_0 = \{r \in R \mid \nu(r) \leq 0\}$ is clearly a subring of R . We have a finite decomposition $R_0 = \bigoplus_{p \in P} R_0 p$ into left ideals and we just need to check that these are simple ideals. Fix some $p \in P$, since $\nu(p) = 0$ we see that $R_0 p$ is a non-zero left ideal. Let $r \neq 0$ be in $R_0 p$ and pick $q \in P$ such that $qr \neq 0$. Now the pair (qr, p) is left P - ν -dependent and, by the weak algorithm, p is left P - ν -dependent on qr , i.e. there exists $s \in R_0 q$ such that $\nu(p - sqr) < \nu(p) = 0$. Thus $sqr = p$ and $R_0 p$ is simple. \square

Definition 3.11. Let (R, ν) be a filtered ring with a set of vertices P and let (M, μ) be a filtered R -module. A subset \mathcal{B} of $\cup_{p \in P} pM$ will be called a *weak P - μ -basis* for M provided that

- (i) Every element in M is left P - μ -dependent on \mathcal{B} .
- (ii) No element of \mathcal{B} is left P - μ -dependent on the rest of \mathcal{B} .

It is easily seen, using the well-ordering of the range of μ , that a weak P - μ -basis of M generates M as an R -module; but in general it need be neither P - μ -independent nor a minimal generating set. However if M satisfies the weak algorithm relative to μ and P then every weak P - μ -basis of M is left P - μ -independent by condition (ii) and hence, by Proposition 3.4, the module M is projective.

The remaining results in this section work in a more general setting but we will state them only for the path algebra, which is the case that we are interested in. From now on E will be a finite quiver with $E^0 = \{1, \dots, d\}$, ν will denote the usual degree in the path algebra and $P = \{p_1, \dots, p_d\}$ will be the natural set of vertices of the path algebra. We can assure existence of weak P - μ -basis for filtered $P(E)$ -modules:

Proposition 3.12. *Let (M, μ) be a filtered $P(E)$ -module. Then there exist sets $\mathcal{B}_h^i \subseteq p_i M_h \setminus M_{h-1}$, for all $i = 1, \dots, d$ and $h \in \mathbb{N}$, such that $\mathcal{B} = \cup_{i,h} \mathcal{B}_h^i$ is a weak P - μ -basis for M . Moreover, the cardinality of \mathcal{B}_h^i does not depend on the weak P - μ -basis.*

Proof. The additive subgroup $M_h = \{m \in M \mid \mu(m) \leq h\}$ has an structure of k^d -module induced by the inclusion $k^d \subseteq P(E)$. For $h > 0$ we denote by M'_h the set of elements in M_h left P - μ -dependent on the set M_{h-1} and put $M'_0 = \{0\}$. Observe that M'_h is also a k^d -module. Indeed, it is clear that M'_h is closed under left product by elements in k^d ; closure with respect to the sum is clear if it has degree h and, otherwise it belongs to M_{h-1} . So, we may consider the k^d -module M_h/M'_h and the set $p_i(M_h/M'_h)$ is a k -vector space. Now, for every $h \geq 0$ and $i = 1, \dots, d$ we pick $\mathcal{B}_h^i \subseteq M_h$ a set of representatives for a k -basis of $p_i(M_h/M'_h)$ such that $\mathcal{B}_h^i \subseteq p_i M_h$. We write $\mathcal{B} = \cup_{i,h} \mathcal{B}_h^i$.

We will show that \mathcal{B} is a weak P - μ -basis for M . By induction on h every element in M_h is left P - μ -dependent on \mathcal{B} . Indeed, for $h = 0$ this holds by construction. Assume that the statement is true for $h \geq 0$. By construction, every element in M_{h+1} differs in some element in M'_{h+1} from a k^d -linear combination of elements in \mathcal{B} (of degree $h+1$). Every element in M'_{h+1} is P - μ -dependent on M_h and every element in M_h is P - μ -dependent on \mathcal{B} . Therefore every element in M_{h+1} is P - μ -dependent on \mathcal{B} . Moreover, since $M = \cup_h M_h$, every element in M is P - μ -dependent on \mathcal{B} .

Suppose that there is $b \in \mathcal{B}$ left P - μ -dependent on $\mathcal{B} \setminus \{b\}$. We write $h = \mu(b)$ and let $p_j \in P$ be such that $p_j b = b$. By construction $b \neq 0$, and hence there exist $(b_i)_{i \in I} \in \prod_{i \in I} (p_{n_i} \mathcal{B} \setminus \{b\})$ and $(r_i)_{i \in I} \in \bigoplus_{i \in I} R p_{n_i}$ such that

$$\mu \left(b - \sum_{i \in I} r_i b_i \right) < h \quad \text{and} \quad \forall i \in I, \nu(r_i) + \mu(b_i) \leq h.$$

Moreover, we can assume that, for all i , $p_j r_i = r_i$. For all i such that $r_i \neq 0$ we have $\mu(b_i) \leq h$ and, if $\mu(b_i) = h$ then $\nu(r_i) = 0$, and so $p_{n_i} = p_j$; therefore b differs in an element in M'_h from a k -linear combination of elements in \mathcal{B}_h^j . This contradicts the fact that classes of elements in \mathcal{B}_h^j are linearly independent elements in $p_j(M_h/M'_h)$. Thus, we get that \mathcal{B} is a weak P - μ -basis for M .

On the other hand, given a weak P - μ -basis \mathcal{C} for M it is clear that classes modulo M'_h of elements in the set $\{c \in \mathcal{C} \mid p_i c = c, \mu(c) = h\}$ give a k -basis of the k -vector space $p_i(M_h/M'_h)$; hence, its cardinality does not depend on the weak P - μ -basis. \square

Now we can characterize projective $P(E)$ -modules using the weak algorithm:

Theorem 3.13. *A $P(E)$ -module M is projective if and only if M satisfies the weak algorithm relative to a suitable filtration.*

Proof. Let M be a projective $P(E)$ -module. Then M is a submodule of some free $P(E)$ -module, say F . By Proposition 3.8, the free module F satisfies the weak algorithm relative to some filtration μ (and P). Therefore, by Lemma 3.9, the module M satisfies the weak algorithm relative to the restriction $\mu|_M$.

Let (M, μ) be a filtered module satisfying the weak algorithm relative to μ and P . By Proposition 3.12, the module M has a weak P - μ -basis, which is P - μ -independent due to the weak algorithm. Hence, by Proposition 3.4, the module M is projective. \square

Let R be a ring and M an R -module. Recall that M is *finitely related* provided that there is an exact sequence of R -modules

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is a free module and L is finitely generated.

The following result generalizes a Theorem by Lewin [22, Theorem 2]. The idea of the proof lies on an unpublished demonstration of Lewin's result due to Warren Dicks [16]. We gratefully acknowledge him for providing it to us.

Theorem 3.14. *Let L be a finitely related $P(E)$ -module. Then L contains a projective module Q such that L/Q has finite k -dimension.*

Proof. Let

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\varphi} L \longrightarrow 0$$

be a presentation for L , where M is free on a subset \mathcal{E} , say, and N is a finitely generated submodule of M . Moreover, since $P(E)$ is a hereditary ring, N is a projective module. It is well-known (see e.g. [4, Proposition 1.2]) that N is isomorphic to a direct sum of copies of the modules $P(E)p_i$; hence, there exists $(f_1, \dots, f_m) \in \prod_{i=1}^m p_{n_i} M$ such that $P(E)f_i \cong P(E)p_{n_i}$ and

$$N = \bigoplus_{i=1}^m P(E)f_i \cong \bigoplus_{i=1}^m P(E)p_{n_i}.$$

We write $\mathcal{F} = \{f_1, \dots, f_m\}$.

Elements in \mathcal{F} are $P(E)$ -linear combinations of elements in \mathcal{E} . Consider a finite subset $\mathcal{E}' \subseteq \mathcal{E}$ such that expressions of elements in \mathcal{F} only involve elements in \mathcal{E}' . Now we define $\mu(\mathcal{E}') = 1$ and extend μ to \mathcal{F} as the formal degree determined by μ and ν , the degree in $P(E)$. We write $n = \max\{\mu(f) \mid f \in \mathcal{F}\}$, define $\mu(\mathcal{E} \setminus \mathcal{E}') = n + 1$ and extend μ to M as

the formal degree. By Proposition 3.8 we get that (M, μ) satisfies the weak algorithm with respect to μ and P .

From Lemma 3.9, N also satisfies the weak algorithm with respect to $\mu' = \mu|_M$ and, by Proposition 3.12, N has a weak P - μ' -basis, say \mathcal{F}' . Therefore, \mathcal{F}' is left P - μ' -independent. Moreover, since N is finitely generated and (by definition of μ') P - μ' -dependent on N_n , \mathcal{F}' is finite and contained in N_n .

Now, we will construct a P - μ -independent family in M in such a way that it gives rise to a projective submodule in L . We have the filtration μ'' on L determined by setting $L_h = (M_h + N)/N$ (viewing L as M/N). Let L'_h denote the set of elements of L_h which are μ'' -dependent on L_{h-1} . For $t > n$ and $i \in E^0$, let \mathcal{B}_t^i be a subset of $p_i M_t$ whose image is a k -basis of $p_i(L_t/L'_t)$. Write $\mathcal{B}^i = \cup_{t>n} \mathcal{B}_t^i$, $\mathcal{B}_t = \cup_{i=1}^d \mathcal{B}_t^i$ and $\mathcal{B} = \cup_{i=1}^d \mathcal{B}^i = \cup_{t>n} \mathcal{B}_t$. Consider the submodule $Q = \sum_{i=1}^d \sum_{b \in \mathcal{B}^i} P(E)\varphi(b) \subseteq L$. The $P(E)$ -module epimorphism defined as follows

$$\bigoplus_{i=1}^d \bigoplus_{b \in \mathcal{B}^i} P(E)p_i \longrightarrow Q$$

$$(r_b^i)_{i,b} \longmapsto \sum_{i=1}^d \sum_{b \in \mathcal{B}^i} r_b^i \varphi(b)$$

is an isomorphism. Indeed, suppose not, then there exist elements $r_b^i \in P(E)p_i$ not all zero such that $\sum_{i=1}^d \sum_{b \in \mathcal{B}^i} r_b^i b \in N$. Therefore there exist elements $r_f \in P(E)p_{n_f}$ satisfying $\sum_{i=1}^d \sum_{b \in \mathcal{B}^i} r_b^i b = \sum_{f \in \mathcal{F}'} r_f f$ (here $p_{n_f} \in P$ is such that $p_{n_f} f = f$). Since $\mathcal{F}' \subseteq N_n$, $\mathcal{B} \cap M_n = \emptyset$ and \mathcal{F}' is P - μ -independent, by the weak algorithm we get an element $b' \in \mathcal{B}' \subseteq \mathcal{B}$ which is P - μ -dependent on $(\mathcal{B} \setminus \{b'\}) \cup \mathcal{F}'$. So, for all i , all $b \in \mathcal{B}^i$ and all $f \in \mathcal{F}'$, there exist elements $s_b^i \in P(E)p_i$, almost all zero, and elements $s_f \in P(E)p_{n_f}$ such that

$$\mu \left(b' - \sum_{b \in \mathcal{B} \setminus \{b'\}} s_b^i b - \sum_{f \in \mathcal{F}'} s_f f \right) < \mu(b')$$

satisfying $\nu(s_b^i) + \mu(b) \leq \mu(b')$ and $\nu(s_f) + \mu(f) \leq \mu(b')$. Moreover, we can assume that $p_{i'} s_b^i = s_b^i$ and $p_{i'} s_f = s_f$. By the same argument used in the proof of Proposition 3.12, we see that $\varphi(\mathcal{B}_{\mu(b')}^{i'})$ is linearly dependent modulo $L'_{\mu(b')}$. This contradicts the fact that the image of $\mathcal{B}_{\mu(b')}^{i'}$ is a k -basis of $p_{i'}(L_{\mu(b'')}/L'_{\mu(b'')})$. Moreover, M_n is finite-dimensional over k and $Q + \varphi(M_n) = L$ so L/Q is finite-dimensional over k . \square

Remark 3.15. Clearly, if L in Theorem 3.14 is finitely presented then Q is also finitely generated.

4. FLATNESS

In this section, we prove that the Leavitt path algebra $L(E)$ is flat as a *right* $P(\overline{E})$ -module. This will play an important role in the sequel. We will denote by $\text{Sink}(E)$ the set of vertices in E which are sinks.

Proposition 4.1. *$L(E)$ is flat as a right $P(\overline{E})$ -module.*

Proof. We write $R = P(\overline{E})$ and $L = L(E)$. To prove that L_R is flat, it suffices to show that $\text{Tor}_1^R(L, M) = 0$ for every left R -module M . We will use the properties of quiver algebras constructed in [4, Section 2]. Recall from there that the Leavitt path algebra is a quotient of $S = (P(E)) \langle \overline{E}; \tau, \delta \rangle$. More exactly, let $X = E^0 \setminus \text{Sink}(E)$ be the set of vertices which are not sinks, then $L = S/I$, where I is the ideal of S generated by the idempotent $q = \sum_{i \in X} p_i - \sum_{e \in E^1} e\overline{e}$ (see [4, Proposition 2.13]). From [4, Proposition 2.5] we know that elements in S can be uniquely written as finite sums $\sum_{\alpha \in E^*} r_\alpha \overline{\alpha}$, where $r_\alpha \in P(E)p_{r(\alpha)}$. On the other hand, elements in $P(E)$ have a unique expression as k -linear combinations of paths. We have that

$$(4.1) \quad S = \bigoplus_{\alpha \in E^*} P(E)\overline{\alpha} = \bigoplus_{\substack{\alpha, \beta \in E^* \\ r(\alpha)=r(\beta)}} k\beta\overline{\alpha} = \bigoplus_{\beta \in E^*} \beta \left(\bigoplus_{\substack{\alpha \in E^* \\ r(\alpha)=r(\beta)}} k\overline{\alpha} \right) = \bigoplus_{\beta \in E^*} \beta R;$$

so, S_R is projective.

Write $q_i = p_i q p_i$. Recall from the proof of (3) in [4, Lemma 2.10] that elements in I can be uniquely written as finite sums

$$\sum_{i \in X} \sum_{\{\alpha \in E^* | r(\alpha)=i\}} r_\alpha q_i \overline{\alpha},$$

where $r_\alpha \in P(E)p_{r(\alpha)}$. Thus, proceeding in the same way as in (4.1) we get that

$$I = \bigoplus_{i \in X} \bigoplus_{\{\gamma \in E^* | r(\gamma)=i\}} \gamma q_i R$$

is projective as right R -module.

Now, the exact sequence of right R -modules

$$0 \longrightarrow I \longrightarrow S \longrightarrow L \longrightarrow 0,$$

gives a projective resolution for L . Let M be a left R -module. We want to see that the induced homomorphism

$$\varphi: \bigoplus_{i \in X} \bigoplus_{\{\gamma \in E^* | r(\gamma)=i\}} \gamma q_i R \otimes_R M \cong I \otimes_R M \longrightarrow S \otimes_R M \cong \bigoplus_{\gamma \in E^*} \gamma R \otimes_R M$$

is a monomorphism. We observe that

$$\varphi \left(\sum_{i \in X} \sum_{\{\gamma \in E^* | r(\gamma)=i\}} \gamma q_i \otimes m_\gamma \right) = \sum_{i \in X} \sum_{\{\gamma \in E^* | r(\gamma)=i\}} \left(\gamma \otimes m_\gamma - \sum_{e \in s^{-1}(i)} \gamma e \otimes \bar{e} m_\gamma \right),$$

and pick a non-zero element

$$x = \sum_{i \in X} \sum_{\{\gamma \in E^* | r(\gamma)=i\}} \gamma q_i \otimes m_\gamma \in \bigoplus_{i \in X} \bigoplus_{\{\gamma \in E^* | r(\gamma)=i\}} \gamma q_i R \otimes M.$$

Let γ_0 be a path of minimum length such that $p_i m_{\gamma_0} \neq 0$, where $i = r(\gamma_0)$. Since $\gamma_0 R \otimes_R M \cong p_i M$, we get $\gamma_0 \otimes m_{\gamma_0} \neq 0$. Note also that the term $\gamma_0 \otimes m_{\gamma_0}$ cannot be cancelled in $\varphi(x)$, because for each of the non-zero terms $\gamma e \otimes \bar{e} m_\gamma$ appearing in that expression, the length of γe is strictly larger than the length of γ_0 , and the sum $\bigoplus_{\gamma \in E^*} \gamma R \otimes_R M$ is a direct sum. It follows that φ is injective and so $\text{Tor}_1^R(L, M) = 0$, as desired. \square

As a consequence, we can regard Leavitt path algebras as perfect left localizations (see [31, Chapter XI]) of path algebras:

Corollary 4.2. *The Leavitt path algebra $L(E)$ is a flat epimorphic left ring of quotients of $P(\bar{E})$.*

Proof. For $i \in E^0 \setminus \text{Sink}(E)$, we write $s^{-1}(i) = \{e_1^i, \dots, e_{n_i}^i\}$ and consider the left $P(\bar{E})$ -module homomorphisms

$$\begin{aligned} \nu_i: \bigoplus_{j=1}^{n_i} P(\bar{E}) p_{s(e_j^i)} &\longrightarrow P(\bar{E}) p_i \\ (r_1, \dots, r_{n_i}) &\longmapsto \sum_{j=1}^{n_i} r_j \bar{e}_j^i. \end{aligned}$$

We write $\Sigma_2 = \{\nu_i \mid i \in E^0 \setminus \text{Sink}(E)\}$ (see the Introduction). It is easy to see that the inclusion $P(\bar{E}) \hookrightarrow L(E)$ is a universal Σ_2 -inverting homomorphism; so, it is a ring epimorphism (see [28, Chapter 4]) and by Proposition 4.1 we get that $L(E)$ is flat as a right $P(\bar{E})$ -module, as desired. \square

Remark 4.3. (1) It is easy to see that the maximal flat epimorphic left ring of quotients of $P(\bar{E})$ is given by the regular algebra of E , i.e. the algebra $Q(E)$ defined in [4], see also the Introduction.

(2) The fact that $L(E)$ is a left quotient ring of $P(\bar{E})$ (equivalently, a right quotient ring of $P(E)$) has been already observed in [30, Proposition 2.2].

5. FINITELY PRESENTED MODULES OVER THE LEAVITT PATH ALGEBRA

Recall that for every left semihereditary ring S , the category of finitely presented left S -modules $\mathbf{fp}(S)$ is an abelian category. (Here, we are looking at $\mathbf{fp}(S)$ as a full subcategory of the category $S\text{-}\mathbf{Mod}$ of all left S -modules. The fact that S is left semihereditary implies that the kernel, the image and the cokernel of every map between finitely presented modules are also finitely presented).

We write $R = P(\overline{E})$ for a finite quiver E , and let \mathcal{T} be the full subcategory of $R\text{-}\mathbf{Mod}$ consisting of all the left R -modules of finite dimension over k . This category is obviously an abelian category, and we will show below that it is the category of objects with finite length in the category $\mathbf{fp}(R)$.

Proposition 5.1. *The category \mathcal{T} of finite-dimensional left R -modules coincides with the category $\mathbf{fp}(R)_{\text{fl}}$ of modules with finite length in $\mathbf{fp}(R)$.*

Proof. First of all, note that every finite-dimensional left R -module is finitely presented by Proposition 2.2. Clearly all the objects in \mathcal{T} are objects of finite length in $\mathbf{fp}(R)$. It remains to see that a simple object in $\mathbf{fp}(R)$ must be finite-dimensional. Let M be a simple object in $\mathbf{fp}(R)$. By Theorem 3.14 (and Remark 3.15), there is a finitely generated projective R -module Q such that $Q \leq M$ and M/Q is finite-dimensional. Since M is simple in $\mathbf{fp}(R)$, we must have $Q = 0$; thus M is finite-dimensional. \square

We write $R = P(\overline{E})$ for some finite quiver E , and A_E for the adjacency matrix of the quiver E .

Proposition 5.2. *Let \mathcal{T} be the category of finite-dimensional left R -modules. Then the following properties hold:*

- (1) $K_0(\mathcal{T})$ is a free abelian group over the set of isomorphism classes of simple, finite-dimensional left R -modules.
- (2) The canonical map $\iota: K_0(R) \rightarrow K_0(\mathbf{fp}(R))$ is an isomorphism, so that $K_0(\mathbf{fp}(R))$ is a free abelian group freely generated by $[Rp_1], \dots, [Rp_d]$.
- (3) The map $K_0(\mathcal{T}) \rightarrow K_0(\mathbf{fp}(R))$ sends $K_0(\mathcal{T})$ onto the subgroup of $K_0(\mathbf{fp}(R))$ generated by the columns of the matrix $\mathbf{1} - A_E^t$.

Proof. (1) Since the category \mathcal{T} coincides with $\mathbf{fp}(R)_{\text{fl}}$ by Proposition 5.1, the result follows from the Devissage Theorem [27, Theorem 3.1.8].

(2) Since R is a left hereditary ring, this is a consequence of the Resolution Theorem [27, Theorem 3.1.13].

(3) We will denote by $[P]$ the class of a projective R -module P in $K_0(R)$ and by $\langle M \rangle$ the class of a finitely presented R -module M in $K_0(\mathbf{fp}(R))$. Moreover, we will identify $K_0(k^d)$ with $K_0(R)$ using the isomorphism induced by the inclusion $k^d \hookrightarrow R$.

Now, let M be a finite-dimensional R -module, by Proposition 2.2 it admits a resolution

$$0 \longrightarrow P \longrightarrow Q \longrightarrow M \longrightarrow 0,$$

where P and Q are finitely generated projective left R -modules. By the identification above and Proposition 2.2 we get the equation

$$\chi_R(M) = (\mathbf{1} - A_E^t) \chi_{k^d}(M)$$

in $K_0(R)$. Moreover, since $\chi_R(M) = [Q] - [P]$ we get

$$\langle M \rangle = \langle Q \rangle - \langle P \rangle = \iota(\chi_R(M)) = (\mathbf{1} - A_E^t) \iota(\chi_{k^d}(M))$$

in $K_0(\mathbf{fp}(R))$. Therefore, the image of $K_0(\mathcal{T})$ is contained in the subgroup generated by the columns of $(\mathbf{1} - A_E^t)$.

To see the reverse inclusion, remember that if $i \in E^0$ is not a source then we have defined the left R -module homomorphisms ν_i . If $i \in E^0$ is a source we define ν_i as the zero homomorphism $0 \rightarrow Rp_i$. Now, the class $\langle \text{coker}(\nu_i) \rangle$ in $K_0(\mathbf{fp}(R))$ coincides with the i -th column of $(\mathbf{1} - A_E^t)$. \square

Let \mathcal{M}_∞ be the full subcategory of $P(\overline{E})\text{-}\mathbf{Mod}$ with objects the modules M such that $L(E) \otimes_{P(\overline{E})} M = 0$. Moreover, we will write \mathcal{M} for the full subcategory of \mathcal{M}_∞ given by its finitely presented modules.

Recall that a *Serre subcategory* of an abelian category \mathcal{A} is an abelian subcategory \mathcal{B} which is closed under subobjects, quotients and extensions. It is easy to see that the kernel of an exact functor between abelian categories is a Serre subcategory (cf. [12, Exercise 6.3.5]), hence the category \mathcal{M}_∞ is a Serre subcategory of $P(\overline{E})\text{-}\mathbf{Mod}$.

Lemma 5.3. *Objects in the category \mathcal{M} are finitely presented $P(\overline{E})$ -modules of finite length. In fact, \mathcal{M} is a Serre subcategory of $\mathbf{fp}(P(\overline{E}))_{\mathfrak{A}}$. Moreover, the induced morphism $K_0(\mathcal{M}) \rightarrow K_0(\mathbf{fp}(P(\overline{E}))_{\mathfrak{A}})$ is a monomorphism and its image is the subgroup generated by the classes of simple modules in \mathcal{M} .*

Proof. Let M be a module in \mathcal{M} . By Theorem 3.14 M has a (finitely generated) projective submodule P of finite codimension, so we have an exact sequence

$$0 \longrightarrow P \longrightarrow M \longrightarrow M/P \longrightarrow 0.$$

Since $L(E)_{P(\overline{E})}$ is flat (Proposition 4.1) we get that $L(E) \otimes_{P(\overline{E})} P = 0$; hence $P = 0$ and M has finite k -dimension. In particular M has finite length.

We have an exact functor $F: \mathbf{fp}(P(\overline{E})) \rightarrow \mathbf{fp}(L(E))$ given by $F(M) = L(E) \otimes_{P(\overline{E})} M$. It follows easily that the kernel of this functor is precisely \mathcal{M} , thus \mathcal{M} is a Serre subcategory of both $\mathbf{fp}(P(\overline{E}))$ and $\mathbf{fp}(P(\overline{E}))_{\mathfrak{A}}$. Now, by the *Devissage* Theorem ([27, Theorem 3.1.8]) we are done. \square

We shall need a result from [23]. We have the following definition:

Definition 5.4 ([23, Definition 0.4]). Let R be a ring and let Σ be a set of homomorphisms of finitely generated projective R -modules. Assume all the maps in Σ are monomorphisms. We define an exact category \mathcal{E} . It is a full subcategory of all R -modules. All objects in \mathcal{E} are finitely presented R -modules, of projective dimension ≤ 1 . The category \mathcal{E} is completely determined by

- (1) For every $s: P \rightarrow Q$ in Σ , the cokernel $M = Q/P$ lies in \mathcal{E} .
- (2) In any short exact sequence of finitely presented R -modules of projective dimension ≤ 1

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

if two of the objects M' , M and M'' lie in \mathcal{E} then so does the third.

- (3) \mathcal{E} contains all direct summands of its objects.
- (4) \mathcal{E} is minimal, subject to (1)–(3).

There is an alternative characterization for this torsion category:

Proposition 5.5 ([23, Proposition 0.7]). *An R -module M belongs to \mathcal{E} if and only if*

- (1) *M is finitely presented, and of projective dimension ≤ 1 .*
- (2) *$R\Sigma^{-1} \otimes_R M = 0 = \text{Tor}_1^R(R\Sigma^{-1}, M)$.*

Following [23], we shall refer to $\mathcal{E} = \mathcal{E}(R, \Sigma)$ as the *category of (R, Σ) -torsion modules*. An object of \mathcal{E} will be a (R, Σ) -torsion module. Using these results, we can characterize the category \mathcal{M} :

Theorem 5.6. *The category \mathcal{M} is the full subcategory of $\mathbf{fp}(P(\overline{E}))_{\mathbb{H}}$ whose objects are the modules having all composition factors in $\{\text{coker } \nu_i \mid i \in E^0 \setminus \text{Sink}(E)\}$.*

Proof. Let M be a module in \mathcal{M} . By definition, the module M is a finitely presented $P(\overline{E})$ -module such that $L(E) \otimes_{P(\overline{E})} M = 0$. Moreover, since $L(E)_{P(\overline{E})}$ is flat (Proposition 4.1) and $P(\overline{E})$ is a hereditary ring the remaining conditions in Proposition 5.5 are fulfilled. Hence we get $\mathcal{M} = \mathcal{E}(P(\overline{E}), \Sigma_2)$ from Proposition 5.5.

Let \mathcal{M}' be the category described in the statement. It is clear that \mathcal{M}' verifies (1)–(4) of Definition 5.4. Thus we get

$$\mathcal{M}' = \mathcal{E}(P(\overline{E}), \Sigma_2) = \mathcal{M},$$

as desired. □

In order to obtain a description of the finitely presented $L(E)$ -modules of finite length we will need the following lemmas (cf. [29, Lemma 6.1]):

Lemma 5.7. *Let N be a finite-dimensional simple $P(\overline{E})$ -module. We have the following dichotomy:*

- (1) *There exist $i \in E^0 \setminus \text{Sink}(E)$ such that $N \cong \text{coker } \nu_i$. In this case $L(E) \otimes_{P(\overline{E})} N = 0$.*
- (2) *For every $i \in E^0 \setminus \text{Sink}(E)$ we have $N \not\cong \text{coker } \nu_i$. In this situation $L(E) \otimes_{P(\overline{E})} N$ is simple.*

Proof. (1) If $N \cong \text{coker } \nu_i$ for some i then $L(E) \otimes_{P(\overline{E})} N = 0$ because $\text{coker } \nu_i \in \mathcal{M}$.

(2) Let N be a finite-dimensional simple left $P(\overline{E})$ -module such that, for every $i \in E^0 \setminus \text{Sink}(E)$, we have $N \not\cong \text{coker } \nu_i$. Theorem 5.6 implies that $N \notin \mathcal{M}$, so that $L(E) \otimes_{P(\overline{E})} N \neq 0$.

Let $n = \sum_{\gamma \in E^*} \gamma \otimes n_\gamma$ be a nonzero element in $L(E) \otimes_{P(\overline{E})} N$, where $n_\gamma \in N$. We may consider the following decomposition of the unit

$$1 = \sum_{i \in E^0} p_i = \sum_{i \notin \text{Sink}(E)} \sum_{e \in s^{-1}(i)} e\bar{e} + \sum_{i \in \text{Sink}(E)} p_i.$$

If $n' := p_i n \neq 0$ for some sink i then $n' \in p_i \otimes p_i N \subseteq 1 \otimes N$. Otherwise, we see that there is some $e \in E^1$ such that $\bar{e}n \neq 0$, and we see inductively that we can find $\gamma \in E^*$ such that $n' := \bar{\gamma}n \neq 0$ and $n' \in 1 \otimes N$. In both cases, the simplicity of N gives us $P(\overline{E})n' = 1 \otimes N$, showing the simplicity of $L(E) \otimes_{P(\overline{E})} N$. \square

Lemma 5.8. *Let i be a vertex. The following are equivalent:*

- (1) $P(\overline{E})p_i$ has finite k -dimension.
- (2) $L(E)p_i$ is a finite direct sum of simple submodules.
- (3) $L(E)p_i$ has finite length.
- (4) The subgraph $s_{E^*}^{-1}(i)$ is acyclic.

Proof. (1) \Rightarrow (2). Let $M \subseteq P(\overline{E})$ be the set of all paths in \overline{E} with range i and starting at a source of \overline{E} . Since $P(\overline{E})p_i$ has finite k -dimension, the set M is finite. We remark that every path in \overline{E} with range i can be extended to a path in M . Now, using the relations $p_j = \sum_{e \in s^{-1}(j)} e\bar{e}$ iteratively and the previous remark we get that $p_i = \sum_{\bar{\gamma} \in M} \gamma\bar{\gamma}$, hence $L(E)p_i = \sum_{\bar{\gamma} \in M} L(E)\gamma\bar{\gamma}$. Moreover, this is a direct sum because elements in the set $\{\gamma\bar{\gamma} \mid \bar{\gamma} \in M\}$ are orthogonal idempotents. On the other hand,

$$L(E)\gamma\bar{\gamma} \cong L(E)\bar{\gamma}\gamma = L(E)p_{r(\gamma)} \cong L(E) \otimes_{P(\overline{E})} P(\overline{E})p_{r(\gamma)}.$$

Since, $r(\gamma) = s(\bar{\gamma})$ is a source in \overline{E} the module $P(\overline{E})p_{r(\gamma)}$ is simple and we are done by Lemma 5.7.

(2) \Rightarrow (3) is obvious.

(3) \Rightarrow (4). Suppose that the subgraph $s_{E^*}^{-1}(i)$ is not acyclic. In particular, there are paths $\alpha, \gamma \in E^*$ such that α is a cycle based at some vertex k , $r(\gamma) = k$ and $s(\gamma) = i$. We write $x = p_i + \gamma\alpha\bar{\gamma}$. If $n > m \geq 1$ are natural numbers then $L(E)x^n \subset L(E)x^m$. Indeed, suppose $y \in L(E)$ is such that $yx^n = x^m$. Since $p_i L(E)p_i \subseteq p_i Q(E)p_i$, operating in the latter ring we get that $y = x^{m-n}$, but $m - n < 0$ and hence $y \notin p_i L(E)p_i$. Therefore, we have constructed an infinite chain of submodules with proper inclusions:

$$L(E)x \supset L(E)x^2 \supset \cdots \supset L(E)x^n \supset \cdots$$

(4) \Rightarrow (1) is clear. \square

Our next result gives a description of the structure of the finitely presented $L(E)$ -modules.

Proposition 5.9. *Let E be a finite quiver and write $R = P(\overline{E})$, $L = L(E)$. Then the following holds:*

- (1) Let N be a finite-dimensional left R -module with a composition series of length k :

$$0 < N_1 < N_2 < \dots < N_k = N.$$

Assume that exactly r composition factors are isomorphic to modules in the set $\{\text{coker } \nu_i \mid i \in E^0 \setminus \text{Sink}(E)\}$. Then $L \otimes_R N$ is a left L -module of finite length and its length is exactly $k - r$.

- (2) Let M be a finitely presented left L -module. Then there is a finitely generated projective L -module P such that $P \leq M$ and M/P is a module of finite length.
- (3) Every finitely presented left L -module M of finite length is isomorphic to a module of the form $L \otimes_R N$, where N is a finite-dimensional left R -module.

Proof. (1) It follows easily from Lemma 5.7 and the fact that L is flat as a right R -module (Proposition 4.1).

(2) Let M be a finitely presented left L -module. By [28, Corollary 4.5] there exists a finitely presented left R -module N such that $L \otimes_R N \cong M$. Now, by Theorem 3.14 (and Remark 3.15), there is a finitely generated projective R -module Q such that $Q \leq N$ and N/Q is finite-dimensional. Since L_R is flat, we have that $M \cong L \otimes_R N$ contains the f.g. projective L -module $P \cong L \otimes_R Q$. By (1), the L -module $(L \otimes_R N)/(L \otimes_R Q) \cong L \otimes_R (N/Q)$ is of finite length.

(3) As above we know that $M \cong L \otimes_R N$ for some finitely presented left R -module N and we obtain (by Theorem 3.14) a projective left R -module Q such that N/Q is finite-dimensional. From the following exact sequence

$$0 \longrightarrow L \otimes_R Q \longrightarrow M \longrightarrow L \otimes_R (N/Q) \longrightarrow 0$$

we get that the projective left L -module $L \otimes_R Q$ has finite length. Since $Q \cong \bigoplus_{i=1}^k R p_{j_i}$ for some $j_i \in E^0$ and every $L \otimes_R R p_{j_i} \cong L p_{j_i}$ has finite length, by Lemma 5.8 we get that every $R p_{j_i}$ is finite-dimensional. Thus, Q is also finite-dimensional, and therefore so is N . \square

6. THE CATEGORY OF FINITELY PRESENTED MODULES AS A QUOTIENT CATEGORY

In this section we will prove that the categories $L(E)\text{-Mod}$, $\mathbf{fp}(L(E))$ and $\mathbf{fp}(L(E))_{\mathbb{A}}$ are equivalent, respectively, to the quotient categories $P(\overline{E})\text{-Mod}/\mathcal{M}_{\infty}$, $\mathbf{fp}(P(\overline{E}))/\mathcal{M}$ and $\mathbf{fp}(P(\overline{E}))_{\mathbb{A}}/\mathcal{M}$. The following results generalize [3, Section 5] to the quiver setting, although quite often the ideas behind the proofs follow [29], where the similar case of the free group algebra is considered.

We first recall some basics on categories. Given a Serre subcategory \mathcal{B} of an abelian category \mathcal{A} , one can consider a quotient abelian category \mathcal{A}/\mathcal{B} and an exact functor $T: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ with the following universal property: given an exact functor $S: \mathcal{A} \rightarrow \mathcal{C}$ from \mathcal{A} to an abelian category \mathcal{C} such that $S(B) \cong 0$ for every object B of \mathcal{B} , there is a unique exact functor $S': \mathcal{A}/\mathcal{B} \rightarrow \mathcal{C}$ such that $S = S'T$ (see [33, Chapter II]). If the category \mathcal{A} is well-powered (that is, every object in \mathcal{A} has a set of representative subobjects) then we can assure the existence of the quotient category \mathcal{A}/\mathcal{B} for any Serre

subcategory \mathcal{B} (see [32, Theorem I.2.1]). Since we only deal with module categories this condition is always fulfilled.

Recall that, given a category \mathcal{C} and a collection Σ of morphisms in \mathcal{C} , the *localization of \mathcal{C} with respect to Σ* is a category \mathcal{C}_Σ , together with a functor $L: \mathcal{C} \rightarrow \mathcal{C}_\Sigma$ such that

- (1) For every $s \in \Sigma$, $L(s)$ is an isomorphism.
- (2) If $F: \mathcal{C} \rightarrow \mathcal{D}$ is any functor sending Σ to isomorphisms in \mathcal{D} , then F factors uniquely through $L: \mathcal{C} \rightarrow \mathcal{C}_\Sigma$.

It turns out that the quotient category \mathcal{A}/\mathcal{B} can also be obtained by localization of \mathcal{A} with respect to the collection of all \mathcal{B} -isos, that is, those maps f such that $\ker(f)$ and $\operatorname{coker}(f)$ are in \mathcal{B} (for details see [33, Appendix in Chapter II]). Thus, we can make use of both universal properties for the quotient category. Moreover, maps in \mathcal{A}/\mathcal{B} are given by equivalence classes $[(f, g)]$ of diagrams in \mathcal{A} ,

$$A_1 \xleftarrow{f} A \xrightarrow{g} A_2$$

where f is a \mathcal{B} -iso.

Let us write $B = L(E) \otimes_{P(\overline{E})} -: P(\overline{E})\text{-Mod} \rightarrow L(E)\text{-Mod}$ for the functor given by extension of scalars and $U: L(E)\text{-Mod} \rightarrow P(\overline{E})\text{-Mod}$ for the functor given by restriction of scalars. We remark that B and U are adjoint functors (see [12, Proposition 3.3.15]). We know that B restricts to a functor between the categories of finitely presented modules and, by Proposition 5.9(1), the same applies to the subcategories of finite length modules. We will also denote these restrictions by B .

Recall from Section 5 that \mathcal{M}_∞ is a Serre subcategory of $P(\overline{E})\text{-Mod}$ and that \mathcal{M} is a Serre subcategory of $\mathbf{fp}(P(\overline{E}))$ and of $\mathbf{fp}(P(\overline{E}))_{\mathfrak{A}}$ (see Lemma 5.3). Therefore, it makes sense to consider the quotient categories $P(\overline{E})\text{-Mod}/\mathcal{M}_\infty$, $\mathbf{fp}(P(\overline{E}))/\mathcal{M}$ and $\mathbf{fp}(P(\overline{E}))_{\mathfrak{A}}/\mathcal{M}$.

Proposition 6.1. *Let $M \in P(\overline{E})\text{-Mod}$ and $N \in L(E)\text{-Mod}$. Then the following properties hold:*

- (1) *There is a natural isomorphism $\eta_N: BU(N) \rightarrow N$.*
- (2) *There is a natural transformation $\theta_M: M \rightarrow UB(M)$.*
- (3) *The composites*

$$\begin{aligned} U(N) &\xrightarrow{\theta_{U(N)}} UB(U(N)) \xrightarrow{U(\eta_N)} U(N) \\ B(M) &\xrightarrow{B(\theta_M)} BUB(M) \xrightarrow{\eta_{B(M)}} B(M) \end{aligned}$$

are identity morphisms.

Proof. (1) Recall that the inclusion $P(\overline{E}) \hookrightarrow L(E)$ is a universal localization; thus it is a ring epimorphism and, by [31, Proposition XI.1.2], the natural transformation $\eta_N: BU(N) \rightarrow N$ defined by $\eta_N(s \otimes n) = sn$ is a natural isomorphism.

(2) It is clear that the homomorphism $\theta_M: M \rightarrow UB(M)$ defined by $\theta_M(m) = 1 \otimes m$ is natural.

(3) It is obvious from the previous definitions. \square

We deduce in the next proposition that B satisfies the same universal property as the localization functor, but only up to natural isomorphism. Let Ξ be the collection of all \mathcal{M}_∞ -isos in $P(\overline{E})\text{-Mod}$.

Proposition 6.2. *If $S: P(\overline{E})\text{-Mod} \rightarrow \mathcal{B}$ is a functor which sends every morphism in Ξ to an isomorphism then there is a functor $S': L(E)\text{-Mod} \rightarrow \mathcal{B}$ such that $S'B$ is naturally isomorphic to S . Moreover, the functor S' is unique up to natural isomorphism.*

Proof. We prove uniqueness first. If there is a natural isomorphism $S \simeq S'B$ then $SU \simeq S'BU \simeq S'$ by Proposition 6.1(1).

To prove existence we must show that if $S' = SU$ then $S'B \simeq S$. Indeed, by Proposition 6.1(3) $B(\theta_M): B(M) \rightarrow BUB(M)$ is an isomorphism for each $M \in P(\overline{E})\text{-Mod}$. Since B is an exact functor (Proposition 4.1) we have $\theta_M \in \Xi$. Thus, $S(\theta): S \rightarrow SUB = S'B$ is a natural isomorphism. \square

Let us consider the localization functor:

$$T: P(\overline{E})\text{-Mod} \rightarrow P(\overline{E})\text{-Mod}/\mathcal{M}_\infty.$$

By the universal property of T there exists a unique functor

$$\overline{B}: P(\overline{E})\text{-Mod}/\mathcal{M}_\infty \longrightarrow L(E)\text{-Mod}$$

such that $B = \overline{B}T$. We will denote by $\mathbf{fp}(P(\overline{E}))_{\mathfrak{fl}}/\mathcal{M}_\infty$ and $\mathbf{fp}(P(\overline{E}))/\mathcal{M}_\infty$ the full subcategories of $P(\overline{E})\text{-Mod}/\mathcal{M}_\infty$ given, respectively, by the finitely presented modules of finite length and by the finitely presented modules. Beware that \mathcal{M}_∞ is not contained in the categories of finitely presented modules so, despite of the notation, these are not quotient categories.

We have the following commutative diagram:

$$\begin{array}{ccccc}
 \mathbf{fp}(P(\overline{E}))_{\mathfrak{fl}} & \xrightarrow{T_{\mathfrak{fl}}} & \frac{\mathbf{fp}(P(\overline{E}))_{\mathfrak{fl}}}{\mathcal{M}_\infty} & \xrightarrow{\overline{B}_{\mathfrak{fl}}} & \mathbf{fp}(L(E))_{\mathfrak{fl}} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{fp}(P(\overline{E})) & \xrightarrow{T_{\mathfrak{fp}}} & \frac{\mathbf{fp}(P(\overline{E}))}{\mathcal{M}_\infty} & \xrightarrow{\overline{B}_{\mathfrak{fp}}} & \mathbf{fp}(L(E)) \\
 \downarrow & & \downarrow & & \downarrow \\
 P(\overline{E})\text{-Mod} & \xrightarrow{T} & \frac{P(\overline{E})\text{-Mod}}{\mathcal{M}_\infty} & \xrightarrow{\overline{B}} & L(E)\text{-Mod} \\
 & & \searrow \text{B} & &
 \end{array}$$

where the vertical arrows are inclusions of full subcategories and the horizontal ones in the first and second rows are given by restriction.

Theorem 6.3. *The functors \overline{B} , $\overline{B}_{\mathfrak{fp}}$ and $\overline{B}_{\mathfrak{fl}}$ are category equivalences.*

Proof. Recall that two categories are equivalent if and only if there is a full, faithful and dense functor between them (see [12, Proposition 1.3.14]). By Proposition 6.2, the functor B satisfies the same natural property than T up to natural isomorphism, hence \overline{B} is a category equivalence. Since \overline{B}_{fp} and \overline{B}_{fl} are given by restriction of \overline{B} , these are full and faithful functors. Moreover, the functor \overline{B}_{fp} is dense by [28, Corollary 4.5] and \overline{B}_{fl} is dense as a consequence of Proposition 5.9(3). \square

Proposition 6.4. *The following holds:*

- (1) *The category $\mathbf{fp}(P(\overline{E}))_{\text{fl}}/\mathcal{M}_{\infty}$ is equivalent to the quotient category $\mathbf{fp}(P(\overline{E}))_{\text{fl}}/\mathcal{M}$.*
- (2) *The category $\mathbf{fp}(P(\overline{E}))/\mathcal{M}_{\infty}$ is equivalent to the quotient category $\mathbf{fp}(P(\overline{E}))/\mathcal{M}$.*

Proof. Let us consider the localization functor in each case:

$$\begin{aligned} S_{\text{fl}}: \mathbf{fp}(P(\overline{E}))_{\text{fl}} &\longrightarrow \mathbf{fp}(P(\overline{E}))_{\text{fl}}/\mathcal{M} \\ S_{\text{fp}}: \mathbf{fp}(P(\overline{E})) &\longrightarrow \mathbf{fp}(P(\overline{E}))/\mathcal{M}. \end{aligned}$$

By the universal property there exist two unique functors

$$\begin{aligned} \overline{T}_{\text{fl}}: \mathbf{fp}(P(\overline{E}))_{\text{fl}}/\mathcal{M} &\longrightarrow \mathbf{fp}(P(\overline{E}))_{\text{fl}}/\mathcal{M}_{\infty} \\ \overline{T}_{\text{fp}}: \mathbf{fp}(P(\overline{E}))/\mathcal{M} &\longrightarrow \mathbf{fp}(P(\overline{E}))/\mathcal{M}_{\infty} \end{aligned}$$

satisfying that $T_{\text{fl}} = \overline{T}_{\text{fl}}S_{\text{fl}}$ and $T_{\text{fp}} = \overline{T}_{\text{fp}}S_{\text{fp}}$. We will show that \overline{T}_{fp} is a full, faithful and dense functor, hence a category equivalence.

Since the categories $\mathbf{fp}(P(\overline{E}))/\mathcal{M}$ and $\mathbf{fp}(P(\overline{E}))/\mathcal{M}_{\infty}$ have the same objects and \overline{T}_{fp} acts as the identity on them it is a dense functor in a trivial way.

Let us write $F = \overline{B}_{\text{fp}}\overline{T}_{\text{fp}}$. The maps in $\mathbf{fp}(P(\overline{E}))/\mathcal{M}$ are equivalence classes $[(f, g)]$ of diagrams in $\mathbf{fp}(P(\overline{E}))$,

$$M_1 \xleftarrow{f} M \xrightarrow{g} M_2$$

where the kernel and the cokernel of f are objects in \mathcal{M} . For such a pair, we have $F([(f, g)]) = (\mathbf{1} \otimes g)(\mathbf{1} \otimes f)^{-1}$. Now assume that $(\mathbf{1} \otimes g)(\mathbf{1} \otimes f)^{-1} = 0$. Then $\mathbf{1} \otimes g = 0$, so $\text{Im}(g) \in \mathcal{M}_{\infty}$. Since $\mathbf{fp}(P(\overline{E}))$ is an abelian category and $\text{Im}(g) = \ker(\text{coker}(g))$ this module is finitely presented and hence in \mathcal{M} . Consequently $[(f, g)] = [(f, 0)] = 0$ and F is a faithful functor. Therefore \overline{T}_{fp} is faithful as well.

Now we will prove that \overline{T}_{fp} is a full functor. Let M_1 and M_2 be finitely presented right $P(\overline{E})$ -modules. A map in $\mathbf{fp}(P(\overline{E}))/\mathcal{M}_{\infty}$ is given by an equivalence class $[(f, g)]$ of diagrams in $P(\overline{E})\text{-Mod}$,

$$M_1 \xleftarrow{f} M \xrightarrow{g} M_2$$

where M is a left $P(\overline{E})$ -module and the kernel and the cokernel of f are objects in \mathcal{M}_{∞} . It is enough to show that it is possible to pick a representative of $[(f, g)]$ with M finitely presented.

Let us write $N' = (\ker f) \cap (\ker g)$. From the following commutative diagram:

$$\begin{array}{ccccc} & & M & & \\ & f \swarrow & \downarrow \pi' & \searrow g & \\ M_1 & \xleftarrow{\bar{f}} & M/N' & \xrightarrow{\bar{g}} & M_2 \end{array}$$

we obtain that $[(f, g)] = [(\bar{f}, \bar{g})]$. So we can assume that $f \oplus g: M \rightarrow M_1 \oplus M_2$ is a monomorphism.

We will show that for such an M we have $M \in \mathbf{fp}(P(\overline{E}))$. By Theorem 3.14 (and Remark 3.15) there exist finitely generated and projective submodules $P_1 \subseteq M_1$, $P_2 \subseteq M_2$ such that M_1/P_1 and M_2/P_2 have finite dimension. Let us write $\pi_1: M_1 \rightarrow M_1/P_1$ and $\pi_2: M_2 \rightarrow M_2/P_2$ for the natural projections and consider the module

$$N = (\ker \pi_1 f) \cap (\ker \pi_2 g).$$

We have the following commutative diagram with exact rows:

$$(6.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N \longrightarrow 0 \\ & & \downarrow f' \oplus g' & & \downarrow f \oplus g & & \downarrow f'' \oplus g'' \\ 0 & \longrightarrow & P_1 \oplus P_2 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_1/P_1 \oplus M_2/P_2 \longrightarrow 0, \end{array}$$

where $f' \oplus g'$ is induced by the universal property of the kernel and $f'' \oplus g''$ is induced by the universal property of the cokernel. Observe that the vertical arrows are monomorphisms. Therefore the module N is projective and the module M/N has finite dimension (and by Proposition 5.1 is finitely presented).

Consider a resolution of M/N by finitely generated projective $P(\overline{E})$ -modules:

$$0 \longrightarrow Q \longrightarrow P \longrightarrow M/N \longrightarrow 0.$$

Applying Schanuel Lemma ([19, (5.1)]) to the previous resolution and to the first row in (6.1) we get the following projective resolution of M :

$$0 \longrightarrow Q \longrightarrow N \oplus P \longrightarrow M \longrightarrow 0.$$

We just need to check that $N \oplus P$ is finitely generated. Recall that in a semihereditary ring every projective module is isomorphic to a direct sum of finitely generated ideals (see [2, Theorem]). Thus, we may consider the following decomposition into direct summands $N \oplus P = Q_1 \oplus Q_2$, where $Q \subseteq Q_1$ and Q_1 is a finitely generated projective module. Now $M \cong (Q_1/Q) \oplus Q_2$ decomposes as a direct sum of a projective module and a finitely presented module. We obtain

$$L(E) \otimes_{P(\overline{E})} M_1 \cong L(E) \otimes_{P(\overline{E})} M \cong \left(L(E) \otimes_{P(\overline{E})} (Q_1/Q) \right) \oplus \left(L(E) \otimes_{P(\overline{E})} Q_2 \right).$$

Since the module $L(E) \otimes_{P(\overline{E})} M_1$ is finitely presented, the module $L(E) \otimes_{P(\overline{E})} Q_2$ is finitely presented as well. Now, since Q_2 is projective, we get that Q_2 is finitely generated and

M is finitely presented. Moreover, $\ker(f), \operatorname{coker}(f) \in \mathcal{M}$ and we have seen that the functor \overline{T}_{fp} is full.

The proof for \overline{T}_{fl} is similar, but simpler because $\mathbf{fp}(P(\overline{E}))_{\text{fl}}$ is closed under subobjects. \square

As a consequence of Theorem 6.3 and Proposition 6.4 we obtain:

Corollary 6.5. *The following holds:*

- (1) *The categories $\mathbf{fp}(P(\overline{E}))_{\text{fl}}/\mathcal{M}$ and $\mathbf{fp}(L(E))_{\text{fl}}$ are equivalent.*
- (2) *The categories $\mathbf{fp}(P(\overline{E}))/\mathcal{M}$ and $\mathbf{fp}(L(E))$ are equivalent.*

7. BLANCHFIELD MODULES OVER A QUIVER

Let R be a ring and let Σ be a family of injective homomorphisms between finitely generated projective R -modules. Recall that, by [23, Proposition 2.2], all maps in Σ are injective in case the localization map $R \rightarrow R\Sigma^{-1}$ is injective.

The localization $R \rightarrow R\Sigma^{-1}$ is *stably flat* if $\operatorname{Tor}_i^R(R\Sigma^{-1}, R\Sigma^{-1}) = 0$ for all $i \geq 2$. Observe that if R is left hereditary then every universal localization $R \rightarrow R\Sigma^{-1}$ is stably flat. Moreover by a result of Bergman and Dicks [11, Theorem 5.3], $R\Sigma^{-1}$ is also left hereditary.

Theorem 7.1 (Neeman, Ranicki [24],[25],[23]). *Let $R \rightarrow R\Sigma^{-1}$ be a stably flat universal localization such that all the morphisms in Σ are injective. Then there is an exact sequence in nonnegative K -theory*

$$\cdots \rightarrow K_{i+1}(R) \rightarrow K_{i+1}(R\Sigma^{-1}) \rightarrow K_i(\mathcal{E}(R, \Sigma)) \rightarrow K_i(R) \rightarrow \cdots$$

Following terminology suggested by [26], we call a left module M over $P(E)$ a *Blanchfield module* in case $\operatorname{Tor}_q^{P(E)}(k^d, M) = 0$ for all q , where we see k^d as a right $P(E)$ -module through the augmentation $\epsilon: P(E) \rightarrow k^d$. It is easy to check that M is a Blanchfield module if and only if the natural map

$$\bigoplus_{e \in E^1} p_{r(e)} M \longrightarrow M, \quad (p_{r(e)} m_e) \mapsto \sum_{e \in E^1} e m_e$$

is an isomorphism (see the proof of Proposition 7.3 for details). Note that this is equivalent to saying that $p_i M = 0$ for every $i \in \operatorname{Sink}(E)$ and that all the maps $\bigoplus_{e \in s^{-1}(i)} p_{r(e)} M \longrightarrow p_i M$, for $i \in E^0 \setminus \operatorname{Sink}(E)$, are isomorphisms. It follows that the Blanchfield modules are exactly the left $L(E)$ -modules M such that $p_i M = 0$ for every $i \in \operatorname{Sink}(E)$.

We will denote the full subcategory of $P(E)$ -**Mod** consisting of all the Blanchfield $P(E)$ -modules by $\mathcal{Bla}_{\infty}(P(E))$, and the category of finitely generated Blanchfield $P(E)$ -modules by $\mathcal{Bla}(P(E))$. Let M be a f.g. Blanchfield $P(E)$ -module. A *lattice* in M is a $P(\overline{E})$ -submodule $A \subset M$ such that A is finite dimensional over k and $M = P(E)A$.

For a ring R , denote by $\mathbf{fnp}(R)_{\text{fl}}$ the full subcategory of finitely presented R -modules of finite length without nonzero projective submodules.

Proposition 7.2. (1) *Let M be a left $L(E)$ -module. Then M is a f.g. Blanchfield $P(E)$ -module if and only if $M \in \mathbf{fnp}(L(E))_{\mathfrak{H}}$.*

(2) *Let M be a f.g. Blanchfield $P(E)$ -module. Then M contains a lattice. Moreover a $P(\overline{E})$ -submodule A of M is a lattice if and only if A is finite dimensional and the natural map $L(E) \otimes_{P(\overline{E})} A \rightarrow M$ is an isomorphism. Furthermore, any lattice in M does not contain nonzero projective $P(\overline{E})$ -submodules.*

(3) *Every f.g. Blanchfield $P(E)$ -module contains a smallest lattice.*

Proof. (1) If M is a finitely presented $L(E)$ -module of finite length without nonzero projective submodules then by Proposition 5.9(3) there is a finite dimensional left $P(\overline{E})$ -module N such that $L(E) \otimes_{P(\overline{E})} N \cong M$. Then clearly M is finitely generated as a $P(E)$ -module. If $i \in \text{Sink}(E)$ and $p_i M \neq 0$, then there is a nonzero map $L(E)p_i \rightarrow M$ which is injective because $L(E)p_i$ is simple, contradicting the fact that M does not contain nonzero projective submodules.

The converse follows from (2).

(2) Assume that M is a left $L(E)$ -module which is finitely generated as $P(E)$ -module. Let a_1, \dots, a_r generators of M as a left $P(E)$ -module. Then, for $e \in E^1$,

$$\overline{e}a_i = \sum_k \gamma_{ji}^e a_j$$

where $\gamma_{ji}^e \in P(E)$. Let r be an upper bound for the lengths of the paths involved in the γ_{ji}^e 's. Let A be the k -space generated by λa_j , where $|\lambda| \leq r$. Then $\overline{e}\lambda a_j \in A$, and clearly A is a lattice for M .

If $A \subset M$ is a finite-dimensional $P(\overline{E})$ -submodule and the natural map $L(E) \otimes_{P(\overline{E})} A \rightarrow M$ is an isomorphism, then $M = P(E)A$ and thus A is a lattice in M . Conversely assume that A is a lattice in M . Since $L(E)$ is flat as a right $P(\overline{E})$ -module, the map $L(E) \otimes_{P(\overline{E})} A \rightarrow L(E) \otimes_{P(\overline{E})} M$ is injective. Now the natural map $L(E) \otimes_{P(\overline{E})} M \rightarrow M$ is an isomorphism, because the inclusion $P(\overline{E}) \rightarrow L(E)$ is a ring epimorphism. It follows that the map $L(E) \otimes_{P(\overline{E})} A \rightarrow M$ is injective. Since A is a lattice this map is clearly surjective.

It follows that M is a finitely presented $L(E)$ -module of finite length. If $p_i M = 0$ for every $i \in \text{Sink}(E)$ then M does not have nonzero projective submodules by Lemma 5.8. Observe that this implies that any lattice A of M does not contain nonzero projective $P(\overline{E})$ -submodules.

(3) This follows as in [3, Proposition 4.1(3)], by showing that the intersection of two lattices is a lattice. \square

Let Σ be the set of square matrices over $P(E)$ that are sent to invertible matrices by the augmentation homomorphism $\epsilon: P(E) \rightarrow k^d$. We have $P_{\text{rat}}(E) \cong P(E)\Sigma^{-1}$, see diagram (1.1) and the comments below it. We are now ready to determine the categories of $(P(E), \Sigma)$ -torsion and $(L(E), \Sigma)$ -torsion.

Proposition 7.3. *With the above notation, we have*

$$\mathcal{E}(P(E), \Sigma) = \mathcal{B}la(P(E)) = \mathcal{E}(L(E), \Sigma).$$

Moreover $\mathcal{B}la(P(E))$ is the class of $P(E)$ -modules isomorphic to cokernels of maps in Σ .

Proof. Note that the objects of $\mathcal{B}la(P(E))$ are automatically $L(E)$ -modules, so that it makes sense to compare $\mathcal{B}la(P(E))$ and $\mathcal{E}(L(E), \Sigma)$.

Let us first show that $\mathcal{B}la(P(E)) = \mathcal{E}(P(E), \Sigma)$. The proof follows arguments in [17] and [26, Section 3]; see also [3, Section 6]. We will include most of the details for completeness.

First we show that the class $\mathcal{E}(P(E), \Sigma)$ is exactly the class of Blanchfield $P(E)$ -modules which are finitely presented as left $P(E)$ -modules. Since $P(E)$ is hereditary, it suffices to show that, for a finitely presented $P(E)$ -module M , we have

$$\mathrm{Tor}_*^{P(E)}(P(E)\Sigma^{-1}, M) = 0 \iff \mathrm{Tor}_*^{P(E)}(k^d, M) = 0.$$

Since M is finitely presented there is an exact sequence

$$(7.1) \quad 0 \longrightarrow P \xrightarrow{d} Q \longrightarrow M \longrightarrow 0$$

with P and Q f.g. projective $P(E)$ -modules. By [7, Remark 3.4], the map $1 \otimes d: P(E)\Sigma^{-1} \otimes_{P(E)} P \rightarrow P(E)\Sigma^{-1} \otimes_{P(E)} Q$ is an isomorphism if and only if the map $\epsilon(d) := 1 \otimes d: k^d \otimes_{P(E)} P \rightarrow k^d \otimes_{P(E)} Q$ is an isomorphism.

For a module X , we use the canonical projective resolution of k^d

$$0 \longrightarrow \bigoplus_{e \in E^1} p_{r(e)} P(E) \xrightarrow{(e)} P(E) \longrightarrow k^d \longrightarrow 0$$

to compute the groups $\mathrm{Tor}_*^{P(E)}(k^d, X)$. It follows that X is a Blanchfield $P(E)$ -module if and only if the map $\gamma_X: \bigoplus_{e \in E^1} p_{r(e)} X \rightarrow X$, $\gamma_X((p_{r(e)} x_e)) = \sum e x_e$, is an isomorphism. Now the diagram in the proof of [26, Proposition 3.9(i)] shows that for the f.p. module M with presentation (7.1), we have that γ_M is an isomorphism if and only if $\epsilon(d)$ is an isomorphism. Hence, by the above comments, M is a Blanchfield module if and only if M is a $(P(E), \Sigma)$ -torsion module.

To finish the proof that $\mathcal{E}(P(E), \Sigma) = \mathcal{B}la(P(E))$, we have to show that every f.g. Blanchfield $P(E)$ -module is finitely presented as $P(E)$ -module. For this part, we follow [17, proof of Lemma 4.3].

Let M be a f.g. Blanchfield $P(E)$ -module. Let A be a lattice in M (Proposition 7.2(2)), and consider the $P(E)$ -module endomorphism of the f.g. projective $P(E)$ -module $P(E) \otimes_{k^d} A$:

$$u: P(E) \otimes_{k^d} A \rightarrow P(E) \otimes_{k^d} A, \quad u(\lambda \otimes a) = \lambda \otimes a - \sum_{e \in E^1} \lambda e \otimes \bar{e}a,$$

where $\lambda \in P(E)$ and $a \in A$. Clearly $\epsilon(u) = 1$, and thus $\mathrm{coker}(u) \in \mathcal{E}(P(E), \Sigma)$. So the previous argument gives that $\mathrm{coker}(u) \in \mathcal{B}la(P(E))$. Let $f: P(E) \otimes_{k^d} A \rightarrow M$ be the map given by $f(\lambda \otimes a) = \lambda a$. Since M is a Blanchfield module we have $fu = 0$, and

thus there is a homomorphism $g: \text{coker}(u) \rightarrow M$ given by $g([\lambda \otimes a]) = \lambda a$. The map $\psi: A \rightarrow \text{coker}(u)$, $\psi(a) = [1 \otimes a]$ is $P(\overline{E})$ -linear. Indeed we have, for $e' \in E^1$,

$$\overline{e'}\psi(a) = \overline{e'}[1 \otimes a] = \overline{e'}\left[\sum_{e \in E^1} e \otimes \overline{e}a\right] = \sum_{e \in E^1} \overline{e'}e[1 \otimes \overline{e}a] = [1 \otimes \overline{e'}a] = \psi(\overline{e'}a).$$

We clearly have the identity $\iota = g\psi$, where $\iota: A \rightarrow M$ denotes the inclusion. In particular ψ is injective and so A is isomorphic with the lattice $\psi(A)$ of $\text{coker}(u)$. By Proposition 7.2(2), the maps $1 \otimes \psi: L(E) \otimes_{P(\overline{E})} A \rightarrow \text{coker}(u)$ and $1 \otimes \iota: L(E) \otimes_{P(\overline{E})} A \rightarrow M$ are both isomorphisms, and clearly $1 \otimes \iota = g(1 \otimes \psi)$. It follows that $g = (1 \otimes \iota)(1 \otimes \psi)^{-1}$ is an isomorphism, so that in particular M is finitely presented as a $P(E)$ -module. Moreover, this argument also shows the last statement in the proposition.

Now we will show that $\mathcal{E}(L(E), \Sigma) = \mathcal{B}la(P(E))$. For $M \in \mathcal{B}la(P(E))$ we have a projective resolution

$$0 \longrightarrow P(E)^n \xrightarrow{\sigma} P(E)^n \longrightarrow M \longrightarrow 0,$$

with $\sigma \in \Sigma$. Note that $\sigma: L(E)^n \rightarrow L(E)^n$ is also injective because the universal localization $L(E) \rightarrow Q(E) = L(E)\Sigma^{-1}$ is injective. Thus we get a resolution of $L(E) \otimes_{P(E)} M$:

$$(7.2) \quad 0 \longrightarrow L(E)^n \xrightarrow{\sigma} L(E)^n \longrightarrow L(E) \otimes_{P(E)} M \longrightarrow 0.$$

Being M an $L(E)$ -module, we get $L(E) \otimes_{P(E)} M \cong M$, and thus $M \in \mathcal{E}(L(E), \Sigma)$.

Now it is straightforward to show that $\mathcal{B}la(P(E)) = \mathcal{E}(P(E), \Sigma)$ satisfies (1)–(4) in Definition 5.4 for the pair $(L(E), \Sigma)$, hence we get $\mathcal{B}la(P(E)) = \mathcal{E}(L(E), \Sigma)$, as desired. \square

In our concluding result we compute the K -groups of the regular algebra $Q(E)$. The Grothendieck group $K_0(Q(E))$ was computed in [4, Theorem 4.2]. We write $\text{Bla}_*(P(E)) = K_*(\mathcal{B}la(P(E)))$ for the K -groups of the exact category $\mathcal{B}la(P(E))$. As a preparation we compute $K_i(P_{\text{rat}}(E))$.

Lemma 7.4. *Let E be a finite quiver with $|E^0| = d$. Then there is a split exact sequence, for $i \geq 1$,*

$$(7.3) \quad 0 \longrightarrow K_i(P(E)) \longrightarrow K_i(P(E)\Sigma^{-1}) \longrightarrow \text{Bla}_{i-1}(P(E)) \longrightarrow 0,$$

and so $K_i(P(E)\Sigma^{-1}) = K_i(P_{\text{rat}}(E)) = K_i(k)^d \oplus \text{Bla}_{i-1}(P(E))$.

Proof. Since $P(E)$ is hereditary, we can apply Theorem 7.1 to the universal localization $P(E) \rightarrow P(E)\Sigma^{-1} = P_{\text{rat}}(E)$ to obtain an exact sequence in nonnegative K -theory

$$\cdots \rightarrow K_i(P(E)) \rightarrow K_i(P(E)\Sigma^{-1}) \rightarrow \text{Bla}_{i-1}(P(E)) \rightarrow K_{i-1}(P(E)) \rightarrow \cdots.$$

We first show that the canonical embedding $k^d \rightarrow P(E)$ induces an isomorphism $K_*(k^d) \rightarrow K_*(P(E))$ for $* \geq 0$. This follows from [18, Theorem 3.1], once we observe that $P_k(E)[t] = P_{k[t]}(E)$ is regular coherent in the sense of [18]. The latter assertion follows from [18, Proposition 1.9 and Remark 1.10], by using induction on the number of arrows of E , taking into account that $P_A(E) \cong P_A(E') *_{A^d} P_A(E'')$, where E' and E''

are subquivers of E with the same vertices and such that E^1 is the disjoint union of E'^1 and E''^1 . The basic case is the one in which the quiver E only has one arrow. If this arrow is a loop then $P_{k[t]}(E)$ is clearly regular coherent because the polynomial rings $k[t]$ and $k[t, s]$ are Noetherian regular rings. If the arrow is not a loop then we get a triangular ring over $k[t]$, and this is again Noetherian regular.

Now note that the isomorphism $K_i(P(E)) \rightarrow K_i(k^d)$, which is induced by the augmentation map, factors through $K_i(P(E)\Sigma^{-1})$, and so we see that the map $K_i(P(E)) \rightarrow K_i(P(E)\Sigma^{-1})$ has a retraction and, in particular, it is injective. This shows the result. \square

Theorem 7.5. *Let E be a finite quiver with $|E^0| = d$. Then $Q(E)$ is the universal localization of $P(\overline{E})$ with respect to the set of all monomorphisms between finitely generated projective left $P(\overline{E})$ -modules whose cokernel is finite-dimensional and does not contain nonzero projective modules. Moreover we have, for $i \geq 1$,*

$$K_i(Q(E)) \cong K_i(L(E)) \bigoplus \text{Bla}_{i-1}(P(E)).$$

In particular

$$\begin{aligned} K_1(Q(E)) &\cong \text{coker}(1 - N_E: (k^\times)^{(E^0 \setminus \text{Sink}(E))} \longrightarrow (k^\times)^{(E^0)}) \\ &\bigoplus \ker(1 - N_E: \mathbb{Z}^{(E^0 \setminus \text{Sink}(E))} \longrightarrow \mathbb{Z}^{(E^0)}) \bigoplus \text{Bla}_0(P(E)) \end{aligned}$$

Proof. Let Υ be the class of all monomorphisms between f.g. projective $P(\overline{E})$ -modules whose cokernel is finite-dimensional and does not contain nonzero projective modules. Let Υ' be the class of monomorphisms between f.g. projective $L(E)$ -modules induced by Υ . Since the maps ν_i , for $i \in E^0 \setminus \text{Sink}(E)$ (defined in the Introduction), are in Υ , we see that $P(\overline{E})\Upsilon^{-1} = L(E)\Upsilon'^{-1}$.

By Proposition 7.2, we have that $\mathcal{Bla}(P(E)) \cong \mathbf{fnp}(L(E))_{\text{fin}}$ is exactly the class of cokernels of maps in Υ' . Since $\mathcal{Bla}(P(E)) = \mathcal{E}(L(E), \Sigma)$ by Proposition 7.3, it follows that

$$Q(E) = L(E)\Sigma^{-1} = L(E)\Upsilon'^{-1} = P(\overline{E})\Upsilon^{-1}.$$

This shows the first part of the theorem.

Since both $P(E)$ and $L(E)$ are hereditary, we can apply Theorem 7.1 to the two universal localizations $P(E) \rightarrow P(E)\Sigma^{-1}$ and $L(E) \rightarrow L(E)\Sigma^{-1} = Q(E)$. Comparison of both localization sequences gives, taking into account Lemma 7.4, the following commutative diagram of exact sequences, for $i \geq 1$:

$$(7.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K_i(P(E)) & \longrightarrow & K_i(P_{\text{rat}}(E)) & \longrightarrow & \text{Bla}_{i-1}(P(E)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow = \\ & & K_i(L(E)) & \longrightarrow & K_i(Q(E)) & \longrightarrow & \text{Bla}_{i-1}(P(E)) \end{array}$$

It follows that the map $K_i(Q(E)) \rightarrow \text{Bla}_{i-1}(P(E))$ is surjective, and so we get a short exact sequence, for $i \geq 1$,

$$(7.5) \quad 0 \longrightarrow K_i(L(E)) \longrightarrow K_i(Q(E)) \longrightarrow \text{Bla}_{i-1}(P(E)) \longrightarrow 0$$

Since the exact sequence (7.3) splits, so does the exact sequence (7.5), by (7.4). The formula for $K_1(Q(E))$ follows now from [6]. \square

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